Pergamon

J. Appl. Maths Mechs, Vol. 59, No. 6, pp. 873–877, 1995
Copyright © 1996 Elsevier Science Ltd
Printed in Great Britain. All rights reserved
0021–8928/95 $24.00+0.00

0021–8928(95)00120-4

A COMPARISON OF THREE METHODS OF CONSTRUCTING LYAPUNOV FUNCTIONS†

V. V. RUMYANTSEV

Moscow

(*Received* 7 *April* 1995)

Chetayev's effective method [1] for constructing Lyapunov functions in the form of a set of first integrals of the equations of perturbed motion has been widely used since the 1950s in Russia. In the 1980s the energy–Casimir method [2] was developed in the U.S.A. as well as the energy–momentum method [3], employed for Hamiltonian systems. A comparison of these methods for systems with a finite number of degrees of freedom has shown that the energy–Casimir method is a more complicated version of Chetayev's method, while the energy–momentum method is essentially the Routh–Lyapunov method [4, 5], stated in modern geometrical language. Some examples are considered.

1. Suppose that, for the equations of perturbed motion

$$\dot{x}_i = X_i(x_1,\dots,x_n) \quad (i=1,\dots,n) \tag{1.1}$$

some independent first integrals are known

$$V_s(x_1,\dots,x_n) = \text{const}, \quad \dot{V}_s = 0 \quad (s=1,\dots,m<n) \tag{1.2}$$

where $X_i(x)$ and $V_s(x)$ are analytic functions, and $X_i(0) = V_s(0) = 0$.

Using Chetayev's method the Lyapunov function is constructed in the form [1]

$$V(x) = \sum_{s=1}^{m} \lambda_s V_s(x) + \sum_{r=1}^{k} \mu_r V_r^2(x), \quad 0 \leqslant k \leqslant m \tag{1.3}$$

where λ_s are constants ($\lambda_1 = 1$), chosen so that the sum $V^{(1)}$ of the terms linear in x_i on the right-hand side of (1.3) is identically zero. The remaining undetermined constants λ_s, and also the constants μ_r, are chosen in such a way that the quadratic form $V^{(2)}$ on the right-hand side of (1.3), which takes form

$$V(x) = V^{(2)}(x) + V_*(x) \tag{1.4}$$

where the function $V_*(x)$ has an order of smallness higher than 2, is sign-definite. Then, in the region of values of x_i that are fairly small in absolute value, the function V will also be sign-definite, and besides $\dot{V}_1 \equiv 0$, and by Lyapunov's theorem on stability, the unperturbed motion $x = 0$ will be stable.

The sufficient conditions for stability obtained in this way turn out, in a number of cases, to be identical (apart from the equality sign) with the necessary conditions. It is interesting that to obtain these conditions it is sometimes sufficient to use only some of the known first integrals [6].

Note. 1. It may turn out that the function $V^{(2)}$ is only a sign-constant function, the function $V^{(3)} \equiv 0$, which $V^{(2)} + V^{(4)}$ is a sign-definite function.

2. Chetayev's method can also be used in the case when $\dot{V}_1 \leqslant 0$; this case arises, in particular, when dissipative forces act on the system.

3. Chetayev's method is closely related [7] to the Routh–Lyapunov theorem [5]. It was proved in [8] that for fairly general assumptions, the stability conditions obtained using Chetayev's method are identical with the conditions obtained using the Routh–Lyapunov theorem.

Example 1. We will consider the problem of the stability of the rotation of a Lagrange top around the vertical, on the basis of the solution of which Chetayev [1] proposed his method in which the Euler–Poisson equations are

†*Prikl. Mat. Mekh.* Vol. 59, No. 6, pp. 916–921, 1995.

written in ω_i. γ_i (i = 1, 2, 3) variables, where ω_i are the projections of the angular velocity and γ_i are the cosines of the angles which the principal axes of inertia make with the vertical. For variety and for comparison with Example 2 we will use the variables $m_i = J_i\omega_i$ and γ_i (i = 1, 2, 3), $J_i > 0$ ($J_1 = J_2$). The known first integrals of the Euler–Poisson equations

$$H(\mathbf{m},\boldsymbol{\gamma}) = \frac{1}{2}\mathbf{m}\cdot\boldsymbol{\omega} + \mathrm{Mg}l\gamma_3 = \text{const}, \quad \mathbf{m}\cdot\boldsymbol{\gamma} = \text{const}, \quad \boldsymbol{\gamma}^2 = 1, \quad m_3 = m = const$$

for the solutions $m_1 = m_2 = 0, m_3 = m, \gamma_1 = \gamma_2 = 0, \gamma_3 = 1$ take the following form in perturbed motion

$$V_1 = 2(H - m^2/(2J_3) - \mathrm{Mg}l) = \text{const}, \quad V_2 = (\mathbf{m}\cdot\boldsymbol{\gamma} - m) = \text{const}$$

$$V_3 = \boldsymbol{\gamma}^2 - 1 = 0, \quad V_4 = m_3 - m = \text{const}$$

Assuming $m_3 = m + x, \gamma_3 = 1 + y$ for the perturbed motion and retaining the notation for the remaining variables, we construct the Lyapunov function of the form (1.3)

$$V = V_1 + 2\lambda V_2 - (\mathrm{Mg}l + \lambda m)V_3 - 2\left(\lambda + \frac{m}{J_3}\right)V_4 + \frac{J_3 - J_1}{J_1 J_3}V_4^2 = \frac{1}{J_1}(m_1^2 + m_2^2 + x^2) +$$

$$+ \ 2\lambda(m_1\gamma_1 + m_2\gamma_2 + xy) - (\mathrm{Mg}l + \lambda m)(\gamma_1^2 + \gamma_2^2 + y^2) \tag{1.5}$$

which is the sum of three similar quadratic forms of two variables each. For these to be positive definite it is necessary and sufficient to choose λ so that the following inequality is satisfied

$$\lambda^2 + \frac{1}{J_1}m\lambda + \frac{1}{J_1}\mathrm{Mg}l < 0 \tag{1.6}$$

The latter inequality is possible if the polynomial has two different real roots, i.e. if

$$m^2 > 4J_1\mathrm{Mg}l \tag{1.7}$$

Inequality (1.7) is the sufficient condition for the unperturbed motion to be stable with respect to the variables m_i, γ_i (i = 1, 2, 3). But this motion is always stable with respect to m_3 in view of the existence of the integral m_3 = const, and it will also be stable with respect to γ_3 if it is stable with respect to γ_i (i = 1, 2). Hence, instead of (1.5) we can consider the function $V = V_1 + 2\lambda V_2$, assuming in it that $m_3 = m$ and replacing γ_s using the integral $V_3 = 0$, i.e.

$$\gamma_3 = 1 - \tfrac{1}{2}(\gamma_1^2 + \gamma_2^2) - \tfrac{1}{8}(\gamma_1^4 + \gamma_2^4) + \ldots$$

Then, the quadratic part of the function

$$V = V_1 + 2\lambda V_2 = \frac{1}{J_1}(m_1^2 + m_2^2) + 2\lambda(m_1\gamma_1 + m_2\gamma_2) - (\mathrm{Mg}l + \lambda m)[\gamma_1^2 + \gamma_2^2 + \tfrac{1}{4}(\gamma_1^4 + \gamma_2^4) + \ldots] \tag{1.8}$$

will be positive-definite in m_i, γ_i (i = 1, 2) for condition (1.7).

In the limiting case

$$m^2 = 4J_1\mathrm{Mg}\,l \tag{1.9}$$

when $\lambda = -m/(2I_1)$, the function (1.8), taking the form

$$V = \frac{1}{J_1}[(m_1 - \sqrt{J_1\mathrm{Mg}l}\,\gamma_1)^2 + (m_2 - \sqrt{J_1\mathrm{Mg}l}\,\gamma_2)^2 + \tfrac{1}{4}\mathrm{Mg}l(\gamma_1^4 + \gamma_2^4) + \ldots$$

is positive-definite, which proves the sufficiency of condition (1.9) for the stability of the vertical rotation of a Lagrange top [1].

The instability of the rotation with respect to m_i, γ_i (i = 1, 2) when

$$m^2 < 4J_1\mathrm{Mg}\,l \tag{1.10}$$

is proved by considering the function $W = m_1\gamma_2 - m_2\gamma_1$ and its derivative with respect to time

$$\dot{W} = \mathrm{M}gl(\gamma_1^2+\gamma_2^2)-\frac{m}{J_1}(m_1\gamma_1+m_2\gamma_2)+\frac{1}{J_1}(m_1^2+m_2^2)[1-\tfrac{1}{2}(\gamma_1^2+\gamma_2^2)-\tfrac{1}{8}(\gamma_1^4+\gamma_2^4)+\dots]$$

in view of the equations of motion with $m_3 = m$, which satisfy all the conditions of Chetayev's theorem on instability [1].

2. The algorithm for investigating the stability of the energy–Casimir method consists of the following steps [2].

(A) Suppose that in the phase space P of the variables $x \in \mathbb{R}^n$ the equations of motion of the form (1.1) have a first integral $H(x)$ = const, usually representing the total energy. In many cases P is a Poisson space, i.e. a linear space which allows the Poisson brackets operation $\{\ ,\ \}$ for real functions in P. Equations (1.1) can then be expressed in Hamiltonian form $\dot{F} = \{F, H\}$, where $H(x)$ is the Hamiltonian and $\dot{F}$ is the derivative of the function $F(x)$ with respect to time.

(B) For Eqs (1.1) one finds a sufficiently large family of constant motions, i.e. a collection $C(x)$ such that $dC(x)/dt = 0$ for any smooth solution of Eqs (1.1). A good way to do this is to use the Hamiltonian formalism to find the Casimir functions, i.e. the functions $C(x)$ which Poisson-commute with any function G defined in the phase space of the Hamiltonian system: $\{C, G\} = 0$. Additional functions related to the symmetries of this Hamiltonian can also be found.

(C) Suppose x_e is a point of equilibrium of system (1.1), i.e. $X(x_e) = 0$, the stability of which is of interest. We find all the Casimir functions C with properties such that the function $H_c = H + C$ has a critical point at x_e

$$\delta H_c(x_e) = 0 \tag{2.1}$$

(D) Calculation of the second variation $\delta^2 H_c(x_e)$ and the requirement that it should be sign-definite for a certain Casimir function which satisfies step (C) leads to the conclusion that the solution x_e is stable by Lyapunov's theorem on stability by virtue of the conservation of H_c.

A comparison of Chetayev's method with the energy–Casimir method shows that the latter is a more complicated version of the first. In fact, in the energy–Casimir method the function $V = H_c(x) - H_c(x_e)$, i.e. the function constructed from the constant motions, the first variation of which $\delta H_c = 0$, while $\delta^2 H_c$ is sign-definite, plays the role of the Lyapunov function. But whereas in Chetayev's method the equality $V^{(1)} = 0$ serves to define the constants λ_s, in the energy–Casimir method Eq. (2.1) serves to define the values of the Casimir functions for $x = x_e$, from which one can construct the Casimir functions themselves, and this problem is obviously more complex than determining λ_s. Correspondingly, it is more difficult to establish the conditions for $\delta^2 H_c$ to be sign-definite than for V^2.

The area of application of the energy–Casimir method is much narrower, since it only applies to Hamiltonian systems for which Casimir functions exist. In a number of important examples the Casimir functions cannot be obtained and may not even exist [3].

Example 2. We will present the solution [2] of the problem of the stability of a Lagrange top using the energy–Casimir method.

The Euler–Poisson equations are Hamiltonian with Hamiltonian function $H(\mathbf{m}, \gamma) = 1/2\mathbf{m}\cdot\boldsymbol{\omega} + \mathrm{M}gl\gamma_3$ in a Lie–Poisson structure $\mathbb{R}^3 \times \mathbb{R}^3$ with Poisson bracket

$$\{F,G\}(\mathbf{m},\boldsymbol{\gamma}) = -\mathbf{m}(\nabla_m F\times\nabla_\gamma G)-\boldsymbol{\gamma}(\nabla_m F\times\nabla_\gamma G+\nabla_\gamma F\times\nabla_m G) \tag{2.2}$$

For any smooth function Φ the quantity $C(\mathbf{m}, \boldsymbol{\gamma}) = \Phi(\mathbf{m}\cdot\boldsymbol{\gamma}, |\boldsymbol{\gamma}|^2)$ which is conserved is the Casimir function in the Poisson structure (2.2).

The first variation of the function $H_c = H + \Phi(\mathbf{m}\cdot\boldsymbol{\gamma}, |\boldsymbol{\gamma}|^2) + \phi(m_3)$, where $\phi(m_3)$ is a smooth function, is

$$\delta H_c = (\boldsymbol{\omega}+\dot{\Phi}\boldsymbol{\gamma})\cdot\delta\mathbf{m}+(\mathrm{M}gl\boldsymbol{\chi}+\dot{\Phi}\mathbf{m}+2\Phi'\boldsymbol{\gamma})\cdot\delta\boldsymbol{\gamma}+\phi'\delta m_3$$

The dot and the prime denote differentiation with respect to the first and second arguments of the function $\Phi(\mathbf{m}\cdot\boldsymbol{\gamma}, |\boldsymbol{\gamma}|^2)$, respectively.

From (2.1) for the solution $\mathbf{m}_e = (0, 0, m)$, $\boldsymbol{\gamma}_e = (0, 0, 1)$ we obtain the equations

$$\omega_3+\dot{\Phi}(m,1)+\phi'=0, \quad \mathrm{M}gl+\dot{\Phi}(m,1)m+2\Phi'(m,1)=0, \quad (\omega_3 = m/J_3)$$

whence we obtain the conditions (correcting the printing errors in (3.202) [2])

$$\dot{\Phi}(m,1) = -\left(\frac{m}{J_3} + \phi'(m)\right), \quad 2\Phi'(m,1) = \left(\frac{m}{J_3} + \phi'(m)\right)m - \mathrm{Mg}l \tag{2.3}$$

which connect Φ, ϕ and the equilibrium $\mathbf{m}_e$, γ_e.

Using the notation

$$a = \phi''(m), \quad b = 4\Phi''(m,1), \quad c = \ddot{\Phi}(m,1), \quad d = 2\dot{\Phi}'(m,1) \tag{2.4}$$

the second variation can be represented in the form of the sum of three quadratic forms of two variations each

$$\delta^2 H_c = \frac{1}{J_1}(\delta m_1^2 + \delta m_2^2) + 2\dot{\Phi}(m,1)(\delta m_1 \delta\gamma_1 + \delta m_2 \delta\gamma_2) + 2\Phi'(m,1)(\delta\gamma_1^2 + \delta\gamma_2^2) +$$
$$+\left(\frac{1}{J_3} + a + c\right)\delta m_3^2 + 2(\dot{\Phi}(m,1) + 2mc + d)\delta m_3 \delta\gamma_3 + (2\Phi'(m,1) + b + m^2 c + 2md)\delta\gamma_3^2 \tag{2.5}$$

which will be positive-definite if and only if

$$\frac{2}{J_1}\Phi'(m,1) - \dot{\Phi}^2(m,1) > 0, \quad \frac{1}{J_3} + a + c > 0$$
$$\left(\frac{1}{J_3} + a + c\right)(2\Phi'(m,1) + b + m^2 c + 2md) - (\dot{\Phi}(m,1) + 2mc + d)^2 > 0 \tag{2.6}$$

The last two inequalities will always be satisfied if we choose the numbers (2.4) appropriately, while, taking (2.3) into account, the first takes the form (correcting the printing error in (3.2, D5) [2])

$$1/J_1(me - \mathrm{Mg}\,l) - e^2 > 0 \tag{2.7}$$

where $e = m/J_3 + \phi'(m)$ can have any value by appropriate choice of $\phi'(m)$. Condition (1.7) is necessary and sufficient to satisfy condition (2.7).

Comparing the quadratic form (1.5) and condition (1.6) for it to be positive-definite with $\delta^2 H_c$ and conditions (2.6), we can see that the latter is more complex than the first.

3. The energy–momentum method [3] is closely related to the method of reduction of simplectic manifolds of dynamic systems with symmetries.

Consider a dynamic system with Hamiltonian function $H(q, p)$. Suppose P is a given phase space of the system. We will assume that a symmetry group G of canonical transformations of P into itself exists, which depends on several parameters. The group G defines several first integrals which form a vector-valued conserved quantity $\mathbf{J}(q, p)$ called the mapping of the momentum.

A set of all points of the phase space P is considered in which $\mathbf{J}(q, p)$ has a given value $\boldsymbol{\mu}$. Such compatible manifolds of the level of integrals in phase space will be invariant manifolds of the phase flux on which a subgroup of the symmetry group acts, the invariant manifolds remaining in place. The factor-manifold of the invariant manifold with respect to this subgroup is called the reduced phase space P_μ.

The reduced phase space P_μ inherits the simplectic structure of the initial space P, so that P_μ can be regarded as a new phase space. The dynamic trajectories of the Hamiltonian H in P define the corresponding trajectories in the reduced space P_μ. This new dynamical system is called a reduced system. Fixed points of the reduced system in P_μ are called relative equilibria (or, more correctly, stationary rotations) of the initial system. In general, the bigger the symmetry group G the richer the supply of relative equilibria.

The relative equilibria q_e, p_e are represented by stationary points of the augmented Hamiltonian

$$H_\xi(q, p) = H(q, p) - \boldsymbol{\xi}\cdot\mathbf{J}(q, p) \tag{3.1}$$

defined by the equation

$$\delta H_\xi(q, p) = 0 \tag{3.2}$$

where $\boldsymbol{\xi}$ can be regarded as a Lagrange multiplier.

To establish the stability of the relative equilibria q_e, p_e, we calculate the second variation $\delta^2 H_\xi$ and we calculate the conditions for it to be sign-definite, which are also the sufficient conditions for stability.

When calculating the second variation of the variable Hamiltonian it is suggested that only those variations of q and p should be used which satisfy the linearized constraint equations $\mathbf{J}$ = const, i.e. $(\delta q, \delta p) \in \ker[D\mathbf{J}(q_e, p_e)]$, and should not lie in symmetry directions. They define the space ν of permissible variations. It has also been shown that if the space ν can be split into two specially chosen subspaces ν_{TB} and ν_{BH}, the matrix $(\delta^2 H_\xi)$ is block-diagonalized, i.e. $\delta^2 H_\xi$ and the simplectic structure can be reduced to normal form simultaneously [3].

Obviously, from the mathematical point of view Eq. (3.2) denotes finding the extremum of the function $H(q, p)$ for a given value of the integrals $\mathbf{J}(q, p) = \boldsymbol{\mu}$, while the condition for $\delta^2 H_\xi$ to be sign-definite is that the extremum will be a minimum or a maximum. Consequently, the energy–momentum method is essentially the Routh–Lyapunov method stated in modern geometrical language.

This research was carried out with financial support from the Russian Foundation for Basic Research (93-013-16242) and the International Science Foundation (MAK-300).

REFERENCES

1. CHETAYEV N. G., *The Stability of Motion*. Papers on analytical mechanics. Izd. Akad. Nauk SSSR, Moscow, 1962.
2. HOLM D. D., MARSDEN J. E., RATIU T. S. and WEINSTEIN A., Nonlinear stability of fluid and plasma equilibria. *Phys. Rep*. **123,** 1–2, 116, 1985.
3. MARSDEN J. E., *Lectures on Mechanics*. Cambridge University Press, 1992.
4. ROUTH E. J., *The Advanced Part of a Treatise on the Dynamics of a System of Rigid Bodies*. Macmillan, London, 1884.
5. LYAPUNOV A. M., On the constant helical motions of a solid in a liquid. In *Collected Papers*, Vol. 1. Izd. Akad. Nauk SSSR, Moscow, 1954.
6. RUMYANTSEV V. V., On the rotational stability of a heavy solid with one fixed point in the S. V. Kovalevskii case. *Prikl. Mat. Mekh*. **18,** 4, 457–458, 1954.
7. RUMYANTSEV V. V., *On the Stability of the Steady Motions of Satellites*. Vych. Tsentr Akad. Nauk SSSR, Moscow, 1967.
8. RUBANOVSKII V. V. and STEPANOV S. Ya., On Routh's theorem and Chetayev's method of constructing Lyapunov's function from the integrals of the equations of motion. *Prikl. Mat. Mekh*. **33,** 5, 904–912, 1969.

Translated by R.C.G.